BULETINUL Universităţii Petrol – Gaze din Ploieşti	Vol. LVIII No. 2/2006	25 - 30	Seria Matematică - Informatică - Fizică

On Viscous Dissipation in Newtonian Fluid Flow through an Annular Cross Section Tube

Tudor Boacă

Universitatea Petrol – Gaze din Ploieşti, Bd. Bucureşti 39, Ploieşti, Catedra de Matematică
e-mail: tboaca@upg-ploiesti.ro

Abstract

This paper considers the problem of viscous dissipation in the flow of Newtonian fluid through a tube of annular cross section, with Dirichlet boundary conditions. The solution of the problem is obtained by a series expansion about the complete eigenfunctions system of a Sturm-Liouville problem. Eigenfunctions and eigenvalues of this Sturm-Liouville problem are obtained by Galerkin's method.

Key words: *dissipation, Newtonian fluid, eigenfunction, Galerkin's method*

Introduction

The problem of viscous dissipation in the fluid flow through a tube of annular cross section has many practical applications. An example is oil product transport through ducts; another is the polymer processing [12].

This problem has constituted the object of many researches. Recently, Valko [11] has obtained an approximate solution by means of a combined method which uses the Laplace transform and Galerkin method. Other approaches of the problem have been given in [6], [9], [7].

In [1] we obtained an approximate solution of the problem of viscous dissipation in the case of incompressible fluid flow through a circular cross section tube.

Now we will consider the flow of Newtonian fluid through a tube of annular cross section with Dirichlet boundary conditions . At the entrance of tube the temperature of fluid is T_0. The walls of radius r_1 and r_2, $r_1 < r_2$ are the same temperature. The flow is slow thus we can neglect the heat transfer by conduction in flow direction. At the same time we will consider that the fluid density ρ, specific heat C_p and the heat transfer coefficient k are constant. The flow is related to a polar spatial coordinate system, the Ox axis is along the tube axis, the radial coordinate will be considered to be r and R is the radius of the tube. For the fluid velocity in the cross section we will consider the expression

$$v(r) = v_0 \cdot f(r) = v_0 \cdot \left[\left(\frac{r_2^2}{r_1^2} - 1 \right) \cdot \ln \frac{r}{r_1} - \left(\frac{r^2}{r_1^2} - 1 \right) \ln \frac{r_2}{r_1} \right], \tag{1}$$

where v_0 is the maximum annular velocity.

Given these conditions the energy equation is [4], [11]:

$$\rho C_p v(r)\frac{\partial T}{\partial x} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \mu\left(\frac{\partial v}{\partial r}\right)^2 \tag{2}$$

where μ is the dynamic viscosity of the fluid.

The aim of this article is to establish an approximate solution of equation (2), which verifies certain initial and boundary conditions.

The plan of the article is: in section two we formulate the mathematical problem, section three will contain the algorithm for the determination of eigenvalues and eigenfunctions (for the Sturm-Liouville problem obtained by method of separation of variables) with Galerkin's method [2]; in the last section we will present the approximate solution of the problem and some numerical results.

The Mathematical Problem

We associate to equation (2) the initial condition

$$x = 0\,,\, T = T_0 \tag{3}$$

and the boundary conditions

$$r = r_1\,,\, T = T_0\,,\, (x > 0) \tag{4}$$

$$r = r_2\,,\, T = T_0\,,\, (x > 0)\,. \tag{5}$$

It is suitable to rewrite the equation (2) and the initial and boundary conditions (3), (4), (5) in dimensionless form. With the transformation group

$$\theta = \frac{T\text{-}T_0}{T_0}\,,\, \eta = \frac{r}{r_1}\,,\, \psi = \frac{k}{\rho C_p r_1^2 v_0}x \tag{6}$$

the equation (2) and the boundary conditions (3), (4), (5) become:

$$f(\eta)\frac{\partial \theta}{\partial \psi} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\eta}\right) + M\cdot\left(\frac{\partial f}{\partial \eta}\right)^2\,,\, \eta \in (1, \eta_0)\,,\, \psi > 0\,, \tag{7}$$

$$\psi = 0\,,\, \theta = 0\,, \tag{8}$$

$$\eta = 1\,,\, \theta = 0\,,\, (\psi > 0)\,, \tag{9}$$

$$\eta = \eta_0\,,\, \theta = 0\,,\, (\psi > 0)\,, \tag{10}$$

where $\eta_0 = \frac{r}{r_2}$, $M = \frac{\mu \cdot v_0^2}{k \cdot T_0}$ and $f(\eta) = (\eta_0^2 - 1)\ln\eta - (\eta^2 - 1)\ln\eta_0$.

It is easy to show that a particular solution of equation (7) which verifies conditions (9) and (10) is:

$$\begin{aligned}\theta_1 = M\cdot\Big[& -\frac{1}{4}(\eta^4 - 1)\cdot\ln^2\eta_0 + (\eta^2 - 1)\cdot(\eta_0^2 - 1)\cdot\ln\eta_0 - \frac{1}{2}(\eta_0^2 - 1)^2\cdot\ln^2\eta + \\ & + \frac{1}{4}(3\cdot\eta_0^4 - 4\cdot\eta_0^2 + 1)\cdot\ln\eta_0\cdot\ln\eta - (\eta_0^2 - 1)^2\cdot\ln\eta\,\Big]\end{aligned} \tag{11}$$

The change of function

$$\theta = u + \theta_1 \tag{12}$$

leads to the equation

$$f(\eta)\frac{\partial u}{\partial \psi} = \frac{1}{r}\frac{\partial}{\partial \eta}\left(r\frac{\partial \theta}{\partial \eta}\right). \tag{13}$$

The unknown function u will satisfy the conditions (9) and (10) and the initial condition (8) is replaced by:

$$\psi = 0\,,\; u = -\theta_1\,. \tag{14}$$

The type of equation (13) and boundary conditions (9) and (10) allow us to apply the method of separation of variables in order to determine function u. By this method the function u is obtained under the form:

$$u(\psi,\eta) = \sum_{n=1}^{\infty} c_n \Phi_n(\eta)\exp\left(-\lambda_n^2 \psi\right), \tag{15}$$

where Φ_n and λ_n are the eigenvalues and the eigenfunctions of Sturm-Liouville problem:

$$\frac{\mathrm{d}}{d\eta}\left(\eta\frac{\mathrm{d}\Phi}{\mathrm{d}\eta}\right) + \lambda^2 \cdot \eta \cdot f(\eta) \cdot \Phi = 0\,, \tag{16}$$

$$\eta = 1\,,\; \Phi = 0\,;\; \eta = \eta_0\,,\; \Phi = 0\,. \tag{17}$$

The Application of Galerkin's Method

For the determination of eigenfunctions and eigenvalues of Sturm-Liouville problem (16), (17) we will apply Galerkin's method. For this we consider the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ defined on $H_0^1(1,\eta_0)\times H_0^1(1,\eta_0)$:

$$\begin{aligned} a(u,v) &= -\int_1^{\eta_0}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}u}{\mathrm{d}\eta}\right)\cdot v\cdot \mathrm{d}\eta = \int_1^{\eta_0}\eta\frac{\mathrm{d}u}{\mathrm{d}\eta}\frac{\mathrm{d}v}{\mathrm{d}\eta}\mathrm{d}\eta, \\ \mathrm{b}(u,v) &= \int_1^{\eta_0}\eta\cdot f(\eta)\cdot u\cdot v\cdot \mathrm{d}\eta\,. \end{aligned} \tag{18}$$

We look for the eigenpair (λ,Φ) which satisfies

$$\begin{aligned} &\Phi \in H_0^1(1,\eta_0),\; \Phi \neq 0 \\ &a(\Phi,v) = \lambda^2 \cdot b(\Phi,v),\; (\forall)\, v \in H_0^1(1,\eta_0) \end{aligned} \tag{17$'$}$$

(17 $'$) is called a variational formulation of (17) [3].

We look for the solution of (17 $'$) under the approximate form

$$\Phi(\eta) = \sum_{k=1}^{n} a_k \varphi_k(\eta)\,, \tag{19}$$

where $n \in \mathbf{N}^*$ is the approach level of function Φ and $(\varphi_k)_{k\in\mathbf{N}^*}$ is a complete system of functions in $L_2[1,\eta_0]$, functions which verify conditions [5]

$$\varphi_k(1) = 0\,,\; \varphi_k(\eta_0) = 0\,,\; k \in \mathbf{N}^*\,. \tag{20}$$

The unknown coefficients $a_k,\; k = \overline{1,n}$ are determined if giving the conditions

$$a\left(\Phi_n,\varphi_j\right)=\lambda^2\cdot b\left(\Phi_n,\varphi_j\right),\ j=\overline{1,n}\,, \tag{21}$$

By applying these conditions we obtain the linear algebraic system in unknown a_k, $k=\overline{1,n}$:

$$\sum_{k=1}^{n}\left(\alpha_{kj}+\lambda^2\beta_{kj}\right)a_k=0\,,\ j=\overline{1,n}\,, \tag{22}$$

where

$$\alpha_{kj}=-a\left(\varphi_k,\varphi_j\right),\ j,k=\overline{1,n}\,, \tag{23}$$

$$\beta_{kj}=b\left(\varphi_k,\varphi_j\right),\ j,k=\overline{1,n}\,. \tag{24}$$

Because the system (22) must have nontrivial solutions, we obtain the equation

$$\Delta_n\equiv\left|\,A+\lambda^2B\,\right|=0\,, \tag{25}$$

where A and B are the matrix $A=\left(\alpha_{kj}\right)_{k,j=\overline{1,n}}$, $B=\left(\beta_{kj}\right)_{k,j=\overline{1,n}}$.

The solutions of equations (25) represent the approximate values, for the n approach level, for the eigenvalues $\lambda_1^2,\ \lambda_2^2,\cdots,\lambda_n^2$.

The solutions of equation (25) are difficult to be obtained under this form. Consequently, through elementary transformations of determinant Δ_n this equation takes the form [8]:

$$\left|\,C-\lambda^2I_n\,\right|=0\,, \tag{26}$$

where I_n is the identity matrix of n order.

Unlike matrix A and B which are symmetric, matrix C does not have this property anymore. Therefore we must adopt an adequate method for the determination of its eigenvalues [13].

In the following we will use the complete system of functions $\left(\varphi_k\right)_{k\in\mathbf{N}^*}$ in $L_2\left[1,\eta_0\right]$:

$$\varphi_k(\eta)=J_0\left(\mu_k\cdot\eta\right)\cdot Y_0\left(\mu_k\right)-J_0\left(\mu_k\right)\cdot Y_0\left(\mu_k\cdot\eta\right), \tag{27}$$

where J_0 and Y_0 are the Bessel function of the first and second kind and zero order respectively and $\mu_k, k\in N^*$ are the roots of equation:

$$J_0\left(\mu\cdot\eta_0\right)\cdot Y_0(\mu)-J_0(\mu)\cdot Y_0\left(\mu\cdot\eta_0\right). \tag{28}$$

The integrals which appear in formulas (23), (24) are calculated with a quadrature formula that must be compatible with Galerkin's method [10]. The eigenvalues of the Sturm-Liouville problem obtained by this method are presented in the next section.

The eigenfunctions of the problem (18), (19) are the analytical form

$$\Phi_i(\eta)=\sum_{j=1}^{n}c_{ij}\left[J_0\left(\mu_k\cdot\eta\right)\cdot Y_0\left(\mu_k\right)-J_0\left(\mu_k\right)\cdot Y_0\left(\mu_k\cdot\eta\right)\right]\,,\ i=\overline{1,n} \tag{29}$$

where $\left(c_{i1},c_{i2},\cdots,c_{in}\right), i=\overline{1,n}$ are the eigenvecteurs of matrix $A+\lambda^2B$.

The Approximate Solution of the Problem

The unknown function u, for the n level of approximation of Galerkin's method, is obtained from (15) and (27):

$$u(\psi,\eta)=\sum_{k=1}^{n}\left(\sum_{i=1}^{n}c_i c_{ik}\,\mathrm{e}^{-\lambda_i^2\psi}\right)\cdot\left[J_0(\mu_k\cdot\eta)\cdot Y_0(\mu_k)-J_0(\mu_k)\cdot Y_0(\mu_k\cdot\eta)\right], \tag{30}$$

The coefficients c_i , $i=\overline{1,n}$ from (30) are determined by the use of the condition (14) and by considering that the solutions Φ_i , $i=\overline{1,n}$ of the problem (16), (17) are orthogonal with weight $\eta\cdot f(\eta)$ on $[1,\eta_0]$ [5]. Because functions Φ_i , $i=\overline{1,n}$ are not obtained exactly, we prefer to use orthogonality with weight η of functions φ_j , $j=\overline{1,n}$ on $[1,\eta_0]$.Thus, for the n level of approximation, the constants c_i , $i=\overline{1,n}$ are determined by the resolution of the linear algebraic system:

$$\sum_{i=1}^{n}c_{ik}c_i=-\frac{\int_0^1\eta\cdot\theta_1(\eta)(\eta)\cdot\left[J_0(\mu_k\cdot\eta)\cdot Y_0(\mu_k)-J_0(\mu_k)\cdot Y_0(\mu_k\cdot\eta)\right]\mathrm{d}\eta}{\int_0^1\eta\cdot\left[J_0(\mu_k\cdot\eta)\cdot Y_0(\mu_k)-J_0(\mu_k)\cdot Y_0(\mu_k\cdot\eta)\right]^2\mathrm{d}\eta},\quad k=\overline{1,n} \tag{31}$$

The final solution of the problem is obtained now by using relations (12), (15) and (30):

$$\begin{aligned}\theta(\psi,\eta)=M\cdot\Big[&-\frac{1}{4}(\eta^4-1)\cdot\ln^2\eta_0+(\eta^2-1)\cdot(\eta_0^2-1)\cdot\ln\eta_0-\frac{1}{2}(\eta_0^2-1)^2\cdot\ln^2\eta+\\&+\frac{1}{4}(3\cdot\eta_0^4-4\cdot\eta_0^2+1)\cdot\ln\eta_0\cdot\ln\eta-(\eta_0^2-1)^2\cdot\ln\eta\Big]+\\&+\sum_{k=1}^{n}\left(\sum_{i=1}^{n}c_i c_{ik}\,\mathrm{e}^{-\lambda_i^2\psi}\right)\cdot\left[J_0(\mu_k\cdot\eta)\cdot Y_0(\mu_k)-J_0(\mu_k)\cdot Y_0(\mu_k\cdot\eta)\right].\end{aligned} \tag{32}$$

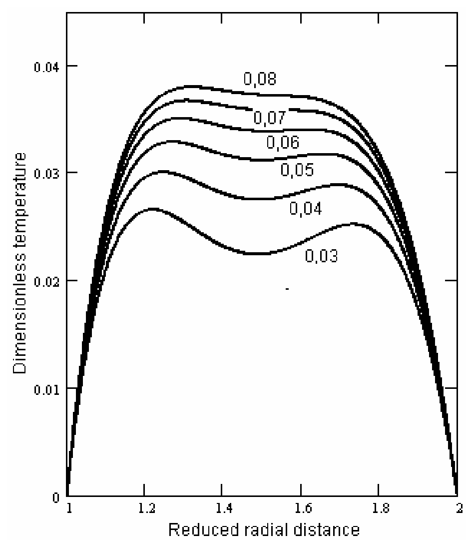

Fig. 1. Dimensionless temperature profiles for M=1, $\eta_0=2$, $\psi=0{,}03...0{,}08$

As an example we will consider $\eta_0=2$ and a fluid with $M=1$. The eigenvalues of Sturm-Liouville problem (16), (17) are presented in table 1. The variation of dimensionless temperature θ given by (32) is presented in figure 1. In abscise axis there is the reduced radial distance η and in axis of ordinates there is presented the dimensionless temperature θ. The variation of dimensionless temperature θ is presented for some values of dimensionless variable ψ.

Table 1. Eigenvalues of Sturm-Liouville problem

n	1	2	3	4	5	6	7	8	9	10
λ_n^2	5,64	12,36	19,09	25,83	32,57	39,31	46,05	52,70	59,53	66,27

The calculations have been realized for the approximation level $n = 10$ and the algorithm presents considerable stability.

As compared to Valko [11], the paper presents the advantage of a simpler algorithm which can also be adapted to other boundary conditions (Dirichlet and Robin type conditions) by an appropriate changing of the condition (17) and of the equation (28).

References

1. Boacă, T. - On Viscous Dissipation in Incompressible Fluid Flow Through a Circular Cross Section Tube, *The Annals of University "Dunărea de Jos" of Galați*, *Fascicle VIII, Tribology* (to appear)
2. Boacă, T. - Utilizarea metodei lui Galerkin pentru determinarea valorilor proprii ale problemei Graetz-Nusselt, *St. Cerc. Mec. Apl.*, 53, 6, pp. 549-560, 1994
3. Ciarlet, P.C., Lions, J.L. - *Handbook of numerical analysis*, vol. II, Elsevier, 1991
4. Constantinescu, V.N. - *Dinamica fluidelor vâscoase în regim laminar,* Ed. Academiei, Bucureşti, 1987
5. Dincă, G. - *Metode variaționale şi aplicații,* Ed. Tehnică, Bucureşti, 1980
6. Gottifredi, J.C., Quiroga, O.D., Floree, A.F. - Heat Transfer to Newtonian and non-Newtonian Fluids flowing in a Laminar Regime, *Int. J. Heat Mass Transfer*, 26, pp. 1215-1220, 1983
7. Johnston, P.R. - A Solution Method for the Graetz Problem for non-Newtonian Fluids with Dirichlet and Neumann Boundary Conditions, *Math. Comput. Model.*, 19, pp. 1-19, 1994
8. Oroveanu, T., Siro, B. - Un algorithme servant à résoudre le problème de Sturm-Liouville par la méthode de Galerkine dans le cas de l'utilisation des systèmes automatiques de calcul, *Rev. Roum. Sci. Techn.-Méc. Appl.*, 30, 2-3, pp. 161-172, 1985
9. Shih, Y.P., Tsou, J.D. - Extended Leveque Solutions for Heat Transfer to Power-law Fluids in Laminar Flow in a Pipe, *Chem. Eng. J.*, 15, pp. 55-62, 1978
10. Şchiop, Al.I. - *Analiza unor metode de* discretizare, Edit. Academiei, Bucureşti, 1978
11. Valkó, P.P. - Solution of the Graetz–Brinkman Problem with the Laplace Transform Galerkin Method, *Int J. Heat Mass Transfer*, 48, pp. 1874-1882, 2005
12. Ybarra, R.M., Eckert, R.E. - Viscous Heat Generation in Slit Flow, *AiChe Journal*, 26, 5, pp. 751-762, 1980
13. William, H.P. - *Numerical Recipes in Pascal,* Cambridge University Press, Cambridge, 1989

Asupra disipației vâscoase în mişcarea unui fluid newtonian printr-un tub de secțiune inelară

Rezumat

În acest articol este studiată problema disipației vâscoase in mişcarea unui fluid newtonian printr-un tub de secțiune inelară, cu condiții la limită de tip Dirichlet. Soluția problemei este obținută sub forma unei serii după sistemul complet de funcții proprii al unei probleme de tip Sturm-Liouville. Valorile proprii şi funcțiile proprii ale acestei probleme Sturm-Liouville sunt obținute cu metoda lui Galerkin.